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ON THE THREE-DIMENSIONAL BENDING THEORY OF CRACKED PLATES

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Abstract—Using an integral formulation, the three-dimensional flexure problem of an elastic plate bounded by the surfaces $z = \pm h$ and containing a through-the-thickness finite crack is reduced to a dual integral equation, which in turn is solved for the unknown function. The analysis shows that, in the interior of the plate, only the stresses σ_{xx}, σ_{yy}, σ_{zz} and τ_{xy} are singular of the order (1/2) and that the angular distribution is precisely the same as that of the corresponding stretching problem. Furthermore, the stress intensity factor is shown to be a function of z as well as a function of the crack to thickness ratio (c/h). The two-dimensional solution is also reduced as the Poisson's ratio $\nu \to 0$.

1. INTRODUCTION

It is well known that the classical theory of bending of thin elastic plates permits the satisfaction of fewer boundary conditions along the edges of a plate than can in reality be prescribed. For instance, along a free edge, only two boundary conditions may be enforced, i.e. the simultaneous vanishing of the bending couple and the equivalent shear (Kirchhoff, 1850). A more refined, yet still approximate theory was introduced by Reissner (1945) who, assuming that the bending stresses are distributed linearly over the thickness of the plate and using Castigliano's theorem of least work, was able to obtain a system of equations which take into account also the transverse shear deformability of the plate. This theory is free of some of the limitations of the classical theory for it permits the satisfaction of three boundary conditions along the edge of the plate.

Applications of Reissner's bending theory have been made to certain classical problems including that of the stress concentration at a circular hole in an infinite plate under flexure and torsion. It was found that, for a wide range of values of the ratio of the diameter of the hole to the thickness of the plate, the numerical results differ considerably from those obtained in the classical theory. However, as Reissner remarks, an exact estimate of the accuracy of the numerical results is not possible unless the exact three-dimensional solution is known.

In view of some recent developments (Folias, 1975), the author in this paper† discusses the bending of the elastic plates from a three-dimensional point of view. Specifically, Navier's equations for the bending of a plate containing a through-the-thickness plane crack are solved.

Considerations of the bending of thin plates containing cracks have been reported by Williams (1957) and Knowles and Wang (1960). In the former reference the bending has been studied on the basis of classical bending theory, and in the latter on the basis of Reissner bending theory.

2. FORMULATION OF THE PROBLEM

Consider the equilibrium of a homogeneous, isotropic, linear elastic plate which occupies the space $|x| < \infty$, $|y| < \infty$, $|z| \leqslant h$, and contains a plane crack in the x–z-plane. The crack faces, defined by $|x| < c$, $y = 0^{\pm}$, $z \leqslant h$ and the plate faces $|z| = h$ are free of stress

† The contents of this paper are based on a report which the author completed in 1975.

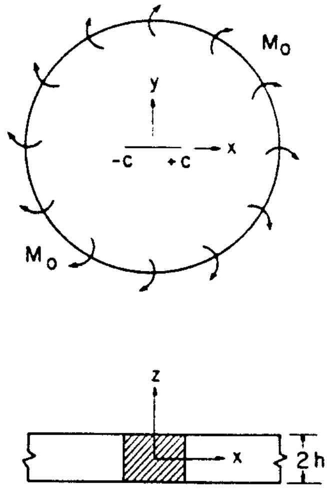

Fig. 1. Geometrical and loading configuration.

and constraint (see Fig. 1). The plate is to be loaded by a constant bending movement M_0 at points far away from the crack.

In the absence of body forces, the coupled differential equations governing the displacement functions u, v and w are:

$$\frac{m}{m-2}\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\theta+\nabla^2(u, v, w) = 0 \tag{1)-(3}$$

where

$$\theta \equiv \frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}, \tag{4}$$

∇^2 is the Laplacian operator, $m \equiv 1/\nu$ and ν is Poisson's ratio.

The stress–displacement relations are given by Hooke's Law as:

$$\sigma_{xx} = 2G\left[\frac{\partial u}{\partial x}+\frac{\theta}{m-2}\right], \ldots, \quad \tau_{xy} = G\left[\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right], \ldots \tag{5)-(10}$$

As to boundary conditions, one must require that at

$$|x| < c, \quad y = 0^{\pm}, \quad |z| \leqslant h: \quad \tau_{xy} = \tau_{yz} = \sigma_{yy} = 0 \tag{11}$$

$$|z| = h: \quad \tau_{xz} = \tau_{yz} = \sigma_{zz} = 0 \tag{12}$$

$$r \to \infty: \quad \tau_{r\phi} = \tau_{rz} = 0, \quad \sigma_{rr} = \frac{3}{2}\frac{M_0}{h^3}z. \tag{13}$$

In addition to the boundary conditions (11)–(13) it is required that u, v, w and all their partial derivatives be continuous for all x, y and $|z| \leqslant h$ except for the points on the plane $-c < x < c$, $y = 0$, $|z| \leqslant h$.

It is found convenient at this point to seek the solution to the crack plate problem in the form

$$u = u^{(P)} + u^{(c)} \text{ etc.}$$

where the first component represents the usual 'undisturbed' or 'particular' solution of a plate without the presence of a crack. Such a particular solution may easily be constructed and for the particular problem at hand is

$$u^{(P)} = \frac{3}{4}\frac{m-1}{m+1}\frac{M_0}{h^3 G}xz$$

$$v^{(P)} = \frac{3}{4}\frac{m-1}{m+1}\frac{M_0}{h^3 G}yz$$

$$w^{(P)} = -\frac{3}{8}\frac{m-1}{m+1}\frac{M_0}{h^3 G}(x^2+y^2) - \frac{3}{4}\frac{1}{m+1}\frac{M_0}{h^3 G}z^2 \tag{14}$$

3. MATHEMATICAL STATEMENT OF THE COMPLEMENTARY PROBLEM

In view of eqn (14) we need to find three functions $u^{(c)}(x, y, z)$, $v^{(c)}(x, y, z)$ and $w^{(c)}(x, y, z)$, such that they satisfy the partial differential equations (1)–(3) and the following boundary conditions:

$$\text{at } |x| < c, \quad y = 0^{\pm}, \quad |z| \leqslant h: \quad \tau_{xy}^{(c)} = \tau_{yz}^{(c)} = 0, \quad \sigma_{yy}^{(c)} = -\frac{3}{2}\frac{M_0}{h^3}z; \tag{15}$$

$$\text{at } |z| = h: \quad \tau_{xz}^{(c)} = \tau_{yz}^{(c)} = \sigma_{zz}^{(c)} = 0; \tag{16}$$

$$\text{at } |y| = 0^{\pm} \text{ and all } x: \quad \tau_{xy}^{(c)} = \tau_{yz}^{(c)} = 0; \tag{17}$$

$$\text{at } r \to \infty: \quad u^{(c)}, v^{(c)} \text{ and } w^{(c)} \text{ must vanish.} \tag{18}$$

4. METHOD OF SOLUTION

In constructing a solution to the system (1)–(3) we use the method introduced by Lur'e (1964), which leads to the following ordinary differential equations with respect to the independent variable z.

$$\frac{d^2 u^{(c)}}{dz^2} + \left(D^2 + \frac{m}{m-2}\partial_1^2\right)u^{(c)} + \left(\frac{m}{m-2}\partial_1\partial_2\right)v^{(c)} + \left(\frac{m}{m-2}\partial_1\right)\frac{dw^{(c)}}{dz} = 0 \tag{19}$$

$$\frac{d^2 v^{(c)}}{dz^2} + \left(D^2 + \frac{m}{m-2}\partial_2^2\right)v^{(c)} + \left(\frac{m}{m-2}\partial_1\partial_2\right)u^{(c)} + \left(\frac{m}{m-2}\partial_2\right)\frac{dw^{(c)}}{dz} = 0 \tag{20}$$

$$\frac{2(m-1)}{m-2}\frac{d^2 w^{(c)}}{dz^2} + \left(\frac{m}{m-2}\partial_1\right)\frac{du^{(c)}}{dz} + \left(\frac{m}{m-2}\partial_2\right)\frac{dv^{(c)}}{dz} + D^2 w^{(c)} = 0, \tag{21}$$

where the symbols of differentiation

$$\partial_1 \equiv \frac{\partial}{\partial x}, \quad \partial_2 \equiv \frac{\partial}{\partial y}, \quad D^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$$

are to be interpreted as numbers. Integrating the above system subject to the values which the complementary displacements attain on the middle plane $z = 0$, i.e.

$$u^{(c)} = 0, \quad v^{(c)} = 0, \quad w^{(c)} = w_0, \quad \frac{du^{(c)}}{dz} = u'_0, \quad \frac{dv^{(c)}}{dz} = v'_0, \quad \frac{dw^{(c)}}{dz} = 0, \quad z = 0, \tag{22}$$

after some simple calculations one has

$$u^{(c)} = \frac{\sin(zD)}{D} u'_0 - \frac{m}{4(m-1)} \left[\frac{\sin(zD)}{D^3} - \frac{z\cos(zD)}{D^2} \right] \partial_1 \theta'_0 \tag{23}$$

$$v^{(c)} = \frac{\sin(zD)}{D} v'_0 - \frac{m}{4(m-1)} \left[\frac{\sin(zD)}{D^3} - \frac{z\cos(zD)}{D^2} \right] \partial_2 \theta'_0 \tag{24}$$

$$w^{(c)} = \cos(zD) w_0 - \frac{m}{4(m-1)} \frac{z\sin(zD)}{D} \theta'_0 \tag{25}$$

where

$$\theta'_0 = \partial_1 u'_0 + \partial_2 v'_0 - D^2 w_0. \tag{26}$$

It may be noted that u'_0, v'_0 and w_0 are arbitrary functions of x and y to be determined. Moreover, in interpreting the operators $\cos(zD)$, and $(1/D)\sin(zD)$, one must expand them in powers of $(zD)^2$, where the differential operators $(zD)^{2n}$ now act on the unknown functions $u'_0(x, y)$, $v'_0(x, y)$ and $w_0(x, y)$.

Using the expressions (23) (26) and the formulae for Hooke's Law, the stresses on any plane perpendicular to the z-axis become

$$\frac{1}{G}\tau_{zx}^{(c)} = \cos(zD)(u'_0 + \partial_1 w_0) - \frac{m}{2(m-1)} \frac{z\sin(zD)}{D} \partial_1 \theta'_0 \tag{27}$$

$$\frac{1}{G}\tau_{yz}^{(c)} = \cos(zD)(v'_0 + \partial_2 w_0) - \frac{m}{2(m-1)} \frac{z\sin(zD)}{D} \partial_2 \theta'_0 \tag{28}$$

$$\frac{1}{G}\sigma_{zz}^{(c)} = -2D\sin(zD) w_0 - \left[\frac{m-2}{2(m-1)} \frac{\sin(zD)}{D} + \frac{m}{2(m-1)} z\cos(zD) \right] \theta_0 \tag{29}$$

and boundary condition (16), therefore, takes the form

$$\begin{aligned} d_{11} u'_0 + d_{12} v'_0 + d_{13} w_0 &= 0 \\ d_{21} u'_0 + d_{22} v'_0 + d_{23} w_0 &= 0 \\ d_{31} u'_0 + d_{32} v'_0 + d_{33} w_0 &= 0, \end{aligned} \tag{30}$$

where the d_{ik} denote the differential operators

$$d_{11} = \cos(hD) - \frac{mh}{2(m-1)} \frac{\sin(hD)}{D} \partial_1^2$$

$$d_{12} = -\frac{mh}{2(m-1)} \frac{\sin(hD)}{D} \partial_1 \partial_2$$

$$d_{13} = \cos(hD)\partial_1 + \frac{mh}{2(m-1)} \frac{\sin(hD)}{D} D^2 \partial_1$$

$$d_{21} = -\frac{mh}{2(m-1)}\frac{\sin(hD)}{D}\partial_1\partial_2$$

$$d_{22} = \cos(hD) - \frac{mh}{2(m-1)}\frac{\sin(hD)}{D}\partial_2^2$$

$$d_{23} = \cos(hD)\partial_2 + \frac{mh}{2(m-1)}\frac{\sin(hD)}{D}D^2\partial_2$$

$$d_{31} = -\left[\frac{m-2}{2(m-1)}\frac{\sin(hD)}{D} + \frac{mh}{2(m-1)}\cos(hD)\right]\partial_1$$

$$d_{32} = -\left[\frac{m-2}{2(m-1)}\frac{\sin(hD)}{D} + \frac{mh}{2(m-1)}\cos(hD)\right]\partial_2$$

$$d_{33} = \frac{2-3m}{2(m-1)}D\sin(hD) + \frac{mh}{2(m-1)}D^2\cos(hD). \tag{31}$$

Using now the fact that the differential operators ∂_1, ∂_2, D^2 obey the same formal rules of addition and multiplication as numbers, one can deduce that

$$\begin{aligned} u_0' &= \chi_1(x,y) \\ v_0' &= \chi_2(x,y) \\ w_0 &= \chi_3(x,y), \end{aligned} \tag{32}$$

where the unknown functions χ_1, χ_2, χ_3 satisfy the differential relations

$$Q\chi_i = 0, \quad i = 1, 2, 3 \tag{33}$$

and the operator Q is defined as†

$$Q = \begin{vmatrix} d_{11} & d_{12} & d_{13} \\ d_{21} & d_{22} & d_{23} \\ d_{31} & d_{32} & d_{33} \end{vmatrix} = \frac{mh}{m-1}D^2\left[1 - \frac{\sin(2hD)}{(2hD)}\right]\cos(hD). \tag{34}$$

By expanding the operator Q in powers of (hD), one notices that the first term leads to the governing equation of classical bending theory while the inclusion of the second term leads to the well known Reissner bending theory.

It remains, therefore, for us to construct a solution to the differential equations (33), which in turn will give us u_0', v_0' and w_0.

Using a Fourier integral transform, we construct next the following integral representations for u_0', v_0', w_0 which have the proper symmetrical behavior at infinity.‡

$$u_0'(x,y^\pm) = \int_0^\infty \left\{(P_1 + |y|Q_1)\,e^{-s|y|} + \sum_{\nu=1}^{\infty} R_\nu^{(1)}\,e^{-\sqrt{s^2+\beta_\nu^2}|y|} + \sum_{n=0}^{\infty} S_n^{(1)}\,e^{-\sqrt{s^2+\alpha_n^2}|y|}\right\}\sin(xs)\,ds \tag{35}$$

† It may be noted that this is a necessary and sufficient condition for the system (30) to have a solution. The question of completeness will be addressed at a later time.

‡ By direct substitution, one can easily verify that eqns (35)–(37) represent solutions to the differential equation (33). First compute D^2u_0', $Du_0'^*$, etc. and then sum up the resulting series. The question of completeness will be discussed later.

$$v_0'(x, y^{\pm}) = \mp \int_0^{\infty} \left\{ (P_2 + |y| Q_2)\, e^{-s|y|} + \sum_{v=1}^{\infty} R_v^{(2)}\, e^{-\sqrt{s^2+\beta_v^2}|y|} + \sum_{n=0}^{\infty} S_n^{(2)}\, e^{-\sqrt{s^2+\alpha_n^2}|y|} \right\} \cos(xs)\, ds \quad (36)$$

$$w_0(x, y^{\pm}) = \int_0^{\infty} \left\{ (P_3 + |y| Q_3)\, e^{-s|y|} + \sum_{v=1}^{\infty} R_v^{(3)}\, e^{-\sqrt{s^2+\beta_v^2}|y|} + \sum_{n=0}^{\infty} S_n^{(3)}\, e^{-\sqrt{s^2+\alpha_n^2}|y|} \right\} \cos(xs)\, ds \quad (37)$$

where the $\pm$ signs refer to $y > 0$ and $y < 0$ respectively,

$$\alpha_n = \left(\frac{2n+1}{2}\right)\frac{\pi}{h} \quad (n = 0, 1, 2, \ldots)$$

and β_v are the roots of the equation

$$\sin(2\beta_v h) = (2\beta_v h). \quad (38)$$

Equation (38) has an infinite number of complex roots which appear in groups of four, one in each quadrant of the complex plane and only two of each group of four roots are relevant to the present work. These are chosen to be the complex conjugate pairs with positive parts. The only real root $\beta_v = 0$ must be ignored.

Finally, for the expressions (35)–(37) to represent the solution of the system (30) one must furthermore require that the unknown functions P_1, Q_1, $R_v^{(1)}$, $S_n^{(1)}$ etc., must satisfy the additional relations

$$Q_2 = -Q_1, \quad sQ_3 = Q_1, \quad s(P_1 + P_2) = Q_1, \quad P_1 - sP_3 = -\frac{2m}{m-1} h^2 s Q_1 \quad (39)\text{–}(42)$$

$$S_n^{(3)} = 0, \quad s_n^{(2)} \equiv S_n, \quad S_n^{(1)} = -\frac{\sqrt{s^2+\alpha_n^2}}{s} S_n \quad (43)\text{–}(45)$$

$$R_v^{(1)} = R_v, \quad R_v^{(2)} = -\frac{\sqrt{s^2+\beta_v^2}}{s} R_v \quad (46)\text{–}(47)$$

$$\left[1 + \frac{m}{2-3m}\cos^2(\beta_v h)\right] R_v^{(3)} = \frac{m-2}{3m-2}\left[1 + \frac{m}{m-2}\cos^2(\beta_v h)\right]\frac{R_v}{s}. \quad (48)$$

In order to facilitate our subsequent discussion, it is found convenient at this stage to summarize our results. Defining

$$\Gamma_v \equiv \frac{\sqrt{s^2+\beta_v^2}\, R_v}{[3m - 2 - m\cos^2(\beta_v h)]} \cdot \frac{\sin(\beta_v h)}{\beta_v h}, \quad (49)$$

one may now write the complementary displacements and stresses in the forms below.

(i) *Complementary displacements*

$$u^{(c)} = \int_0^\infty \left\{ \left[z(P_1 + |y|Q_1) + \frac{z^3}{3}\left(\frac{2m-1}{m-1}\right) sQ_1 \right] e^{-s|y|} \right.$$
$$+ \sum_{\nu=1}^{\infty} \frac{\Gamma_\nu}{\beta_\nu} \cos(\beta_\nu h)[(2m-2-m\cos^2(\beta_\nu h))\sin(\beta_\nu z) + m\beta_\nu z\cos(\beta_\nu z)]$$
$$\left. \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2+\beta_\nu^2}} - \sum_{n=0}^{\infty} \frac{S_n}{\alpha_n} \sin(\alpha_n z) \frac{\sqrt{s^2+\alpha_n^2}\, e^{-\sqrt{s^2+\alpha_n^2}|y|}}{s} \right\} \sin(xs)\, ds \quad (50)$$

$$v^{(c)} = \mp \int_0^\infty \left\{ \left[z\left(\frac{Q_1}{s} - P_1 - |y|Q_1\right) - \frac{2m-1}{m-1}\frac{z^3}{3} sQ_1 \right] e^{-s|y|} \right.$$
$$- \sum_{\nu=1}^{\infty} \frac{\Gamma_\nu}{s\beta_\nu} \cos(\beta_\nu h)[(2m-2-m\cos^2(\beta_\nu h))\sin(\beta_\nu z) + m\beta_\nu z\cos(\beta_\nu z)]$$
$$\left. \times e^{-\sqrt{s^2+\beta_\nu^2}|y|} + \sum_{n=0}^{\infty} \frac{S_n}{\alpha_n} \sin(\alpha_n z)\, e^{-\sqrt{s^2+\alpha_n^2}|y|} \right\} \cos(xs)\, ds \quad (51)$$

$$w^{(c)} = \int_0^\infty \left\{ \left[\frac{P_1}{s} + \frac{2m}{m-1} h^2 Q_1 - \frac{1}{m-1} z^2 Q_1 + |y| \frac{Q_1}{s} \right] e^{-s|y|} \right.$$
$$+ \sum_{\nu=1}^{\infty} \frac{\Gamma_\nu}{s} \cos(\beta_\nu h)[(m-2+m\cos^2(\beta_\nu h))\cos(\beta_\nu z) + m\beta_\nu z\sin(\beta_\nu z)]$$
$$\left. \times \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2+\beta_\nu^2}} \right\} \cos(xs)\, ds \quad (52)$$

$$\theta^{(c)} = \int_0^\infty \left\{ \frac{2(m-2)}{m-1} zQ_1\, e^{-s|y|} - 2(m-2) \sum_{\nu=1}^{\infty} \frac{\beta_\nu \Gamma_\nu}{s} \cos(\beta_\nu h)\sin(\beta_\nu z) \right.$$
$$\left. \times \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2+\beta_\nu^2}} \right\} \cos(xs)\, ds. \quad (53)$$

(ii) *Complementary stresses*

$$\frac{\sigma_{zz}^{(c)}}{2G} = -m \int_0^\infty \sum_{\nu=1}^{\infty} \frac{\beta_\nu \Gamma_\nu}{s} \cos(\beta_\nu h)\{\cos^2(\beta_\nu h)\sin(\beta_\nu z) - \beta_\nu z\cos(\beta_\nu z)\}$$
$$\times \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2+\beta_\nu^2}} \cos(xs)\, ds \quad (54)$$

$$\frac{\tau_{xz}^{(c)}}{G} = \int_0^\infty \left\{ -\frac{2m}{m-1}(h^2 - z^2) sQ_1\, e^{-s|y|} \right.$$
$$+ 2m \sum_{\nu=1}^{\infty} \Gamma_\nu \cos(\beta_\nu h)[\sin^2(\beta_\nu h)\cos(\beta_\nu z) - \beta_\nu z\sin(\beta_\nu z)]$$
$$\left. \times \frac{e^{-\sqrt{s^2+\beta_\nu^2}|y|}}{\sqrt{s^2+\beta_\nu^2}} - \sum_{n=0}^{\infty} S_n \cos(\alpha_n z) \frac{\sqrt{s^2+\alpha_n^2}}{s} e^{-\sqrt{s^2+\alpha_n^2}|y|} \right\} \sin(xs)\, ds \quad (55)$$

$$\frac{\tau_{yz}^{(c)}}{G} = \mp\int_0^\infty \Bigg\{\frac{2m}{m-1}(h^2-z^2)sQ_1\,e^{-s|y|} -2m\sum_{v=1}^{\infty}\Gamma_v\cos(\beta_v h)[\sin^2(\beta_v h)\cos(\beta_v z)-\beta_v z\sin(\beta_v z)] \times\frac{e^{-\sqrt{s^2+\beta_v^2}|y|}}{s}+\sum_{n=0}^{\infty}S_n\cos(\alpha_n z)\,e^{-\sqrt{s^2+\alpha_n^2}|y|}\Bigg\}\cos(xs)\,ds \quad (56)$$

$$\frac{\tau_{xy}^{(c)}}{G} = \mp\int_0^\infty \Bigg\{\left[2z(sP_1-Q_1)+\frac{2}{3}\left(\frac{2m-1}{m-1}\right)z^3s^2Q_1+2sz|y|Q_1\right]e^{-s|y|} +2\sum_{v=1}^{\infty}\frac{\Gamma_v}{\beta_v}\cos(\beta_v h)[(2m-2-m\cos^2(\beta_v h))\sin(\beta_v z)+m\beta_v z\cos(\beta_v z)] \times e^{-\sqrt{s^2+\beta_v^2}|y|}-\sum_{n=0}^{\infty}\frac{S_n}{\alpha_n}\sin(\alpha_n z)\left(\frac{2s^2+\alpha_n^2}{s}\right)e^{-\sqrt{s^2+\alpha_n^2}|y|}\Bigg\}\sin(xs)\,ds \quad (57)$$

$$\frac{\sigma_{yy}^{(c)}}{2G} = \int_0^\infty \Bigg\{z\left[\frac{2m}{m-1}Q_1-sP_1-\frac{2m-1}{m-1}\frac{z^2s^2}{3}Q_1-|y|sQ_1\right]e^{-s|y|} -2\sum_{v=1}^{\infty}\frac{\Gamma_v\beta_v}{s}\cos(\beta_v h)\sin(\beta_v z)\frac{e^{-\sqrt{s^2+\beta_v^2}|y|}}{\sqrt{s^2+\beta_v^2}} -\sum_{v=1}^{\infty}\frac{\Gamma_v}{s\beta_v}\cos(\beta_v h)[(2m-2-m\cos^2(\beta_v h))\sin(\beta_v z)+m\beta_v z\cos(\beta_v z)] \times\sqrt{s^2+\beta_v^2}\,e^{-\sqrt{s^2+\beta_v^2}|y|}+\sum_{n=0}^{\infty}\frac{S_n}{\alpha_n}\sin(\alpha_n z)\sqrt{s^2+\alpha_n^2}\,e^{-\sqrt{s^2+\alpha_n^2}|y|}\Bigg\}\cos(xs)\,ds \quad (58)$$

$$\frac{\sigma_{xx}^{(c)}}{2G} = \int_0^\infty \Bigg\{z\left[sP_1+\frac{2}{m-1}Q_1+|y|sQ_1+\frac{2m-1}{m-1}\frac{z^2s^2}{3}Q_1\right]e^{-s|y|} -2\sum_{v=1}^{\infty}\frac{\Gamma_v\beta_v}{s}\cos(\beta_v h)\sin(\beta_v z)\frac{e^{-\sqrt{s^2+\beta_v^2}|y|}}{\sqrt{s^2+\beta_v^2}} +\sum_{v=1}^{\infty}\frac{\Gamma_v}{s\beta_v}\cos(\beta_v h)[(2m-2-m\cos^2(\beta_v h))\sin(\beta_v z)+m\beta_v z\cos(\beta_v z)] \times\frac{s^2e^{-\sqrt{s^2+\beta_v^2}|y|}}{\sqrt{s^2+\beta_v^2}}-\sum_{n=0}^{\infty}\frac{S_n}{\alpha_n}\sin(\alpha_n z)\sqrt{s^2+\alpha_n^2}\,e^{-\sqrt{s^2+\alpha_n^2}|y|}\Bigg\}\cos(xs)\,ds. \quad (59)$$

By direct substitution, it can easily be ascertained that the above complementary displacements satisfy Navier's equations and furthermore, the corresponding stresses $\sigma_{zz}^{(c)}$, $\tau_{zz}^{(c)}$, $\tau_{yz}^{(c)}$ do vanish on the plate faces $z = \pm h$. Moreover, for future applications, the solution may be put in a more convenient form, see Appendix B.

It is appropriate at this time to examine the question of completeness of the above eigenfunctions. This matter has been investigated† by Wilcox (1978) who, using a double Fourier integral transform in x and y and the subsequent use of a contour integration, arrived precisely at the same eigenfunctions as those obtained in the above analysis. This established, therefore, the fact that the above eigenfunctions do indeed constitute a complete set.

Finally, if we consider the following two combinations to vanish (sufficient conditions)

† In 1977, the author used the same idea to show that the set of eigenfunctions was indeed complete. Seeking, however, an independent opinion, he posed the question to Prof. Wilcox.

$$2m \sum_{\nu=1}^{\infty} \frac{\Gamma_\nu}{\beta_\nu s} \cos(\beta_\nu h)[-\cos^2(\beta_\nu h)\sin(\beta_\nu z)+\beta_\nu z\cos(\beta_\nu z)]$$

$$-\sum_{n=0}^{\infty} \frac{S_n}{\alpha_n}\sin(\alpha_n z) = \frac{2m}{m-1}\left(h^2 z - \frac{z^3}{3}\right) sQ_1 \quad (60)$$

and

$$2\sum_{\nu=1}^{\infty} \frac{\Gamma_\nu}{\beta_\nu}\cos(\beta_\nu h)[(2m-2-m\cos^2(\beta_\nu h))\sin(\beta_\nu z)+m\beta_\nu z\cos(\beta_\nu z)]$$

$$-\sum_{n=0}^{\infty} \frac{2s^2+\alpha_n^2}{\alpha_n}\frac{S_n}{s}\sin(\alpha_n z) = -2z\left[sP_1 - Q_1 + \frac{2m-1}{m-1}\frac{z^2}{3}s^2 Q_1\right] \quad (61)$$

for all $|z| \leqslant h$, then two of the remaining stress boundary conditions are satisfied automatically, i.e.

$$\tau_{xy}^{(c)} = \tau_{yz}^{(c)} = 0 \quad \text{for all } x, \quad |z| \leqslant h \text{ and } y = 0.$$

We may now combine eqns (60) and (61) to obtain

$$\sum_{\nu=1}^{\infty} \frac{\Gamma_\nu}{\beta_\nu}\cos(\beta_\nu h)[(2m-2+m\cos^2(\beta_\nu h))\sin(\beta_\nu z)-m\beta_\nu z\cos(\beta_\nu z)]$$

$$-m\sum_{\nu=1}^{\infty} \frac{\Gamma_\nu}{s^2}\beta_\nu\cos(\beta_\nu h)[(1+\sin^2(\beta_\nu h))\sin(\beta_\nu z)+\beta_\nu z\cos(\beta_\nu z)]$$

$$= -z\left(sP_1 + \frac{m+1}{m-1}Q_1 + \frac{2m}{m-1}h^2 s^2 Q_1\right) + \frac{1}{m-1}\frac{z^3}{3}s^2 Q_1. \quad (62)$$

A study of eqns (60) and (62) shows that it is convenient at this stage to let†

$$Q_1 = 0 \quad \text{and} \quad sP_1 = 2m\sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{s^2}\beta_\nu^2.$$

Thus, using the Fourier Series expansions given in Appendix B, eqn (60) yields

$$\frac{S_n}{\alpha_n} = \frac{8m}{h}(-1)^n \sum_{\nu=1}^{\infty}\frac{\Gamma_\nu\beta_\nu^2}{s}\frac{\cos^2(\beta_\nu h)}{(\alpha_n^2-\beta_\nu^2)^2}. \quad (63)$$

Returning next to the last boundary condition, we require that

$$\int_0^\infty \sum_{\nu=1}^{\infty}\frac{\Gamma_\nu}{\beta_\nu s^2}\Bigg\{-2\beta_\nu^2\cos(\beta_\nu h)\sin(\beta_\nu z)\frac{s}{\sqrt{s^2+\beta_\nu^2}}$$

$$+\cos(\beta_\nu h)[2m-2-m\cos^2(\beta_\nu h))\sin(\beta_\nu z)+m\beta_\nu z\cos(\beta_\nu z)][s^2-s\sqrt{s^2+\beta_\nu^2}]$$

$$-\frac{8m}{h}\sum_{n=0}^{\infty}(-1)^n\frac{\cos^2(\beta_\nu h)}{(\alpha_n^2-\beta_\nu^2)^2}\sin(\alpha_n z)\left[s^2+\frac{\alpha_n^2}{2}-s\sqrt{s^2+\alpha_n^2}\right]\Bigg\}\cos(xs)\,ds$$

$$= -\left(\frac{3}{4}\right)\frac{M_0}{h^3 G}z; \quad |x| < c, \quad |z| \leqslant h \quad (64)$$

† An easy way to see that $Q_1 = 0$ is to let, in eqn (60), $z = h$ and then use eqn (63) together with the result in Appendix A.

where we have made use of eqns (61) and (63). Furthermore, along $|x| > c$ and $|y| = 0$ we must require that the complementary displacements, together with their first partial derivatives, are continuous for all $|z| < h$. The latter can be accomplished if one considers the following integral combination to vanish:

$$\int_0^\infty \frac{\Gamma_\nu}{s^3} \cos(xs)\, ds = 0; \quad |x| > c. \tag{65}$$

The problem, therefore, has been reduced to that of solving the dual integral equation (64), (65) for the unknown functions Γ_ν subject to the condition (62). But this is plausible for the Γ_νs are complex

5. SOLUTION OF THE INTEGRAL EQUATION

Seeking a solution of the form

$$\frac{\Gamma_\nu}{s^3} = \sum_{k=0}^\infty A_\nu^{(k)} \frac{J_{k+1}(sc)}{(sc)^{k+1}}, \tag{66}$$

and employing the method described in Folias (1975), one finds after some straightforward computations

$$\sum_{\nu=1}^\infty \sum_{k=0}^\infty \frac{A_\nu^{(k)}}{\beta_\nu} \Big\{ -2\beta_\nu^2 \cos(\beta_\nu h) \sin(\beta_\nu z) I_3^{kj}(\beta_\nu) + \cos(\beta_\nu h)[(2m-2-m\cos^2(\beta_\nu h)) \sin(\beta_\nu z)$$
$$+ m\beta_\nu z \cos(\beta_\nu z)] I_2^{kj}(\beta_\nu) - \frac{8m}{h} \sum_{n=0}^\infty (-1)^n \frac{\beta_\nu^3 \cos^2(\beta_\nu h)}{(\alpha_n^2 - \beta_\nu^2)^2} \sin(\alpha_n z) \left[I_2^{kj}(\alpha_n) + \frac{\alpha_n^2}{2} I_1^{kj} \right] \Big\}$$
$$= -\left(\frac{3}{4}\right) \frac{M_0}{h^3 G} z \frac{1}{2^{j+1}(j+1)!}; \quad |z| \leqslant h \quad (j = 0, 1, 2, \ldots) \tag{67}$$

where the integrals I_n^{kj} have been defined as:

$$I_1^{kj} = \int_0^\infty s \frac{J_{k+1}(sc)}{(sc)^{k+1}} \frac{J_{j+1}(sc)}{(sc)^{j+1}}\, ds = \frac{1}{2^{k+j+1} k!\, j!\, (k+j+1) c^2}$$
$$I_2^{kj}(\beta) = \int_0^\infty s\{s^2 - s\sqrt{s^2+\beta^2}\} \frac{J_{k+1}(sc)}{(sc)^{k+1}} \frac{J_{j+1}(sc)}{(sc)^{j+1}}\, ds$$
$$I_3^{kj}(\beta) = \int_0^\infty s \left\{ \frac{s}{\sqrt{s^2+\beta^2}} \right\} \frac{J_{k+1}(sc)}{(sc)^{k+1}} \frac{J_{j+1}(sc)}{(sc)^{j+1}}\, ds. \tag{68}$$

Similarly, eqn (62), in view of the identity

$$z^2 \frac{J_{n+1}(z)}{(z)^{n+1}} = (2n) \frac{J_n(z)}{(z)^n} - \frac{J_{n-1}(z)}{(z)^{n-1}} \tag{69}$$

and eqn (66) yields

$$\sum_{\nu=1}^\infty A_\nu^{(0)} a_\nu(z) = 0 \tag{70a}$$

$$\sum_{\nu=1}^\infty A_\nu^{(1)} a_\nu(z) = 0 \tag{70b}$$

$$\sum_{\nu=1}^{\infty} \{(2k)A_\nu^{(k)}a_\nu(z) - A_\nu^{(k+1)}a_\nu(z) + A_\nu^{(k-1)}(\beta_\nu c)^2 b_\nu(z)\} = 0 \quad (k = 1,2,3,\ldots) \tag{70c}$$

where for convenience we have defined

$$a_\nu(z) \equiv \frac{\cos(\beta_\nu h)}{\beta_\nu}[(2m-2+m\cos^2(\beta_\nu h))\sin(\beta_\nu z) - m\beta_\nu z\cos(\beta_\nu z)] \tag{71}$$

and

$$b_\nu(z) \equiv -m\frac{\cos(\beta_\nu h)}{\beta_\nu}[(1+\sin^2(\beta_\nu h))\sin(\beta_\nu z) + \beta_\nu z\cos(\beta_\nu z)] + 2mz. \tag{72}$$

Prior to engaging ourselves with any numerical computations, it is wise at this point to investigate analytically the behavior of the solution of eqn (70a), through the thickness $|z| \leqslant h$. However, for convenience, we choose to study its derivative with respect to z, i.e.

$$\sum_{\nu=1}^{\infty} A_\nu^{(0)}\cos(\beta_\nu h)[(m-2+m\cos^2(\beta_\nu h))\cos(\beta_\nu z) + m\beta_\nu z\sin(\beta_\nu z)] = 0; \quad |z| < h. \tag{73}$$

Defining

$$f(\xi) = \sum_{\nu=1}^{\infty} A_\nu^{(0)}\cos(\beta_\nu h\xi), \tag{74}$$

the above equation may be written in the form

$$(m-1)[f(1+\zeta)+f(1-\zeta)+\frac{m}{2}(1-\zeta)f'(1+\zeta)+\frac{m}{2}(1+\zeta)f'(1-\zeta) = 0 \tag{75}$$

where

$$\zeta \equiv (z/h).$$

This is a difference-differential equation, the solution of which is

$$f^{(h)}(1+\xi) = (1-\zeta)^{2-2/m}\left\{\sum_{n=0}^{\infty} a_{2n}\frac{\zeta^{2n+2}}{(2n+2)}\,{}_2F_1\left(-3+\frac{2}{m},2n+2;2n+3;-\zeta\right)+C_0\right\} \tag{76a}$$

where ${}_2F_1(\ldots)$ is the hypergeometric function and the constants C_0 and a_{2n} are to be chosen so that eqn (67) is also satisfied. Thus considering the first term only, which represents the dominant part of the solution, one has:

$$2\sum_{\nu=1}^{\infty} A_\nu^{(0)}\cos(\beta_\nu h)\cos(\beta_\nu z) = C_0\left\{\left(1-\frac{z}{h}\right)^{2-2/m}+\left(1+\frac{z}{h}\right)^{2-2/m}\right\}. \tag{76b}$$

Similarly,

$$2\sum_{\nu=1}^{\infty} A_\nu^{(1)}\cos(\beta_\nu h)\cos(\beta_\nu z) = C_1\left\{\left(1-\frac{z}{h}\right)^{2-2/m}+\left(1+\frac{z}{h}\right)^{2-2m}\right\} \tag{77}$$

etc.

The solution, therefore, of eqns (67) and (70a, b, c) will determine the unknown coefficients $A_\nu^{(k)}$ which are functions of (c/h) and Poisson's ratios. However, as we will see later, one needs only to compute the $A_\nu^{(0)}$, for this is the only coefficient that contributes to the highest order singularity. Thus, pending a numerical study of this system, the author in this paper considers the coefficient known.

6. THE STRESS FIELD AHEAD OF THE CRACK

In the vicinity of the crack tip, the stresses may now be expressed in terms of the following four types of integrals:†

$$M_1 = \int_0^\infty J_1(sc)\left\{e^{-s|y|} - \frac{s}{\sqrt{s^2+\beta_\nu^2}}\, e^{-\sqrt{s^2+\beta_\nu^2}|y|}\right\}\cos(xs)\, ds \tag{78}$$

$$M_2 = \int_0^\infty J_1(sc)\left\{\left(s^2+\frac{\beta_\nu^2}{2}-\frac{\beta_\nu^2}{2}|y|s\right)e^{-s|y|} - s\sqrt{s^2+\beta_\nu^2}\, e^{-\sqrt{s^2+\beta_\nu^2}|y|}\right\}\cos(xs)\, ds \tag{79}$$

$$M_3 = \int_0^\infty J_1(sc)\, e^{-s|y|-isx}\, ds = -\sqrt{\frac{c}{2\varepsilon}}\, e^{i(\phi/2)} + 0(\varepsilon^0) \tag{80}$$

$$M_4 = |y|\int_0^\infty sJ_1(sc)\, e^{-s|y|-isx}\, ds = -\frac{1}{4}\sqrt{\frac{c}{2\varepsilon}}\,[e^{i(\phi/2)} - e^{i(5\phi/2)}] + 0(\varepsilon^0). \tag{81}$$

Unfortunately, the first two integrals we have not as yet been able to evaluate in closed form. However, one can show by contour integration that they are of the $0(\varepsilon^0)$ as $y \to \varepsilon \sin\phi$ and $x \to c+\varepsilon\cos\phi$.

In view of the above and by virtue of eqn (76b), the stresses in the interior portion of the plate are found to be:

$$\sigma_{xx}^{(c)} = \sigma_0\left(\frac{\Lambda}{2}\right)\{(1+\zeta)^{1-2\nu}-(1-\zeta)^{1-2\nu}\}\sqrt{\frac{c}{2\varepsilon}}\left[\frac{3}{4}\cos\left(\frac{\phi}{2}\right)+\frac{1}{4}\cos\left(\frac{5\phi}{2}\right)\right]+0(\varepsilon^0) \tag{82}$$

$$\sigma_{yy}^{(c)} = \sigma_0\left(\frac{\Lambda}{2}\right)\{(1+\zeta)^{1-2\nu}-(1-\zeta)^{1-2\nu}\}\sqrt{\frac{c}{2\varepsilon}}\left[\frac{5}{4}\cos\left(\frac{\phi}{2}\right)-\frac{1}{4}\cos\left(\frac{5\phi}{2}\right)\right]+0(\varepsilon^0) \tag{83}$$

$$\sigma_{zz}^{(c)} = \sigma_0\left(\frac{\Lambda}{2}\right)\{(1+\zeta)^{1-2\nu}-(1-\zeta)^{1-2\nu}\}\sqrt{\frac{c}{2\varepsilon}}\left[\nu\cos\left(\frac{\phi}{2}\right)\right]+0(\varepsilon^0) \tag{84}$$

$$\tau_{xy}^{(c)} = \sigma_0\left(\frac{\Lambda}{2}\right)\{(1+\zeta)^{1-2\nu}-(1-\zeta)^{1-2\nu}\}\sqrt{\frac{c}{2\varepsilon}}\left[-\frac{1}{4}\sin\left(\frac{\phi}{2}\right)+\frac{1}{4}\sin\left(\frac{5\phi}{2}\right)\right]+0(\varepsilon^0) \tag{85}$$

$$\tau_{yz}^{(c)} = -\sigma_0\left(\frac{\Lambda}{2}\right)\left(\frac{c}{h}\right)\{(1+\zeta)^{-2\nu}+(1-\zeta)^{-2\nu}\}\sqrt{\frac{\varepsilon}{2c}}\left[\frac{(1-2\nu)}{2}\cos\left(\frac{\phi}{2}\right)\sin(\phi)\right]+0(\varepsilon^0) \tag{86}$$

† The higher order of Bessel terms of eqn (66) contribute to stress terms of $0(\varepsilon^{1/2})$.

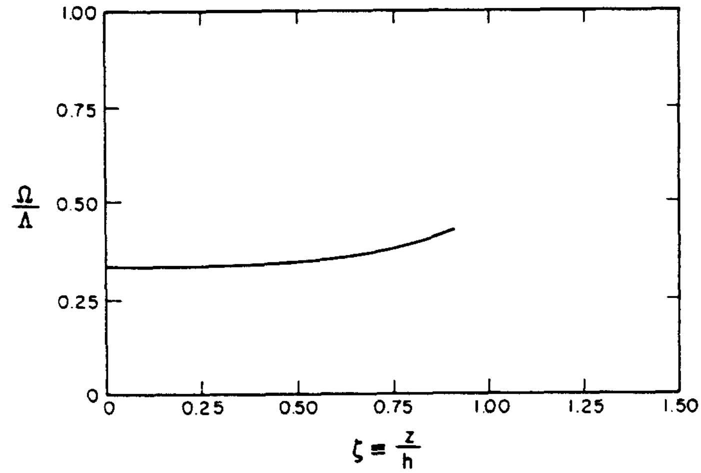

Fig. 2. The behavior of (Ω/Λ) versus ζ and for Poisson's ratio 1/3.

$$\tau_{zx}^{(c)} = \sigma_0\left(\frac{\Lambda}{2}\right)\left(\frac{c}{h}\right)\{(1+\zeta)^{-2\nu}+(1-\zeta)^{-2\nu}\}\sqrt{\frac{\varepsilon}{2c}}\left[(1-2\nu)^2\cos\left(\frac{\phi}{2}\right)\right.$$

$$\left. +\frac{(1-2\nu)}{2}\sin\left(\frac{\phi}{2}\right)\sin(\phi)\right]+0(\varepsilon^0)\quad(-h+\varepsilon^* < z < h-\varepsilon^*),\qquad(87)$$

where for simplicity we have defined

$$\zeta \equiv \frac{z}{h},\quad \sigma_0 \equiv \frac{1+\nu}{3+\nu}\frac{6M_0}{h^2},\quad \Lambda \equiv -\frac{4(1-\nu)}{\nu}\left(\frac{2G}{\sigma_0 ch}\right)C_0.\qquad(88)$$

The reader should notice† that C_0 is proportional to ν and therefore the limit as $\nu \to 0$ of Λ is finite. Furthermore, the constant C_0 is related to $A_\nu^{(0)}$ through the equation

$$\sum_{\nu=1}^{\infty} A_\nu^{(0)} = 2^{2-2/m}C_0.\qquad(89)$$

We observe now that in the interior layers of the plate the angular distribution of the stresses is exactly the same as in the corresponding case of the stretching problem (Folias, 1975). Furthermore, our results differ from those obtained by Knowles and Wang (1960) by a factor of

$$\Omega = \left(\frac{\Lambda}{2\zeta}\right)\{(1+\zeta)^{1-2\nu}-(1-\zeta)^{1-2\nu}\},\qquad(90)$$

which represents the variation across the thickness (see Fig. 2).

Finally, the author would like to emphasize that the stress field (82)–(87) is valid only in the interior of the plate, i.e. for all $-h+\varepsilon^* < z < h-\varepsilon^*$, where ε^* is some distance away from the free surface of the plate.

The reason for this is that eqns (82)–(87) reflect only the dominant part of the solution (76a), i.e. the C_0 term only. This is the price that one must pay in order to obtain a simple but analytical expression. On the other hand, if one considers the remaining terms a_{2n}, he can extend the validity of the stress intensity factor up to a very small boundary layer adjacent to the free edge.

In general, therefore, the value of ε^* depends on the crack to thickness ratio as well as on the Poisson's ratio ν. The author estimates ε^* to be approximately $0.4h$.

† See eqn (67). Because of the factor of m on the left-hand side of eqn (67), it is clear that $A_\nu^{(0)} \sim (1/m)$.

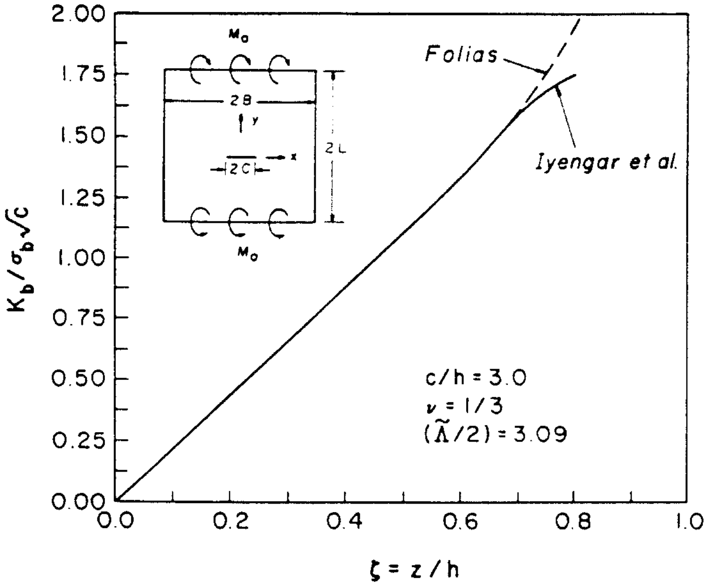

Fig. 3. The stress intensity factor versus z/h for crack to thickness ratio of $c/h = 0.25$.

However, as one approaches the corner point $z = h$, or $\zeta = 1$, then other terms of the $0(\varepsilon^{-1/2+2n}\beta_\nu^{2n-1})$, $n = 0, 1, 2, \ldots$, also contribute to the same order singularity and therefore must be accounted for. Only then can the stresses σ_{zz}, τ_{yz} and τ_{xz} be shown to vanish at the plate faces $z = \pm h$. For the special case of a plate with a circular hole and under the action of a tensile load, this matter has been investigated by Folias and Wang (1985). Extensive numerical work in that case revealed the presence of a boundary layer effect in the neighborhood of the free surface of the plate. Moreover, ananalytical asymptotic solution there shows (see Folias, 1987) the stress field to be proportional to ρ^α, where $\alpha = 1.73959 \pm i\ 1.11902$. This result is identical to that obtained by Williams (1952) for a 90° material corner with free–free of stress boundaries. It is also of interest to note that all stresses, except $\tau_{r\theta}$, exhibit the same order of singularity in the neighborhood of the corner point.

Returning next to eqns (82)–(87), the value of Λ can easily be calculated from eqns (67) and (70). Without going into the details, the stress intensity factor across the thickness,

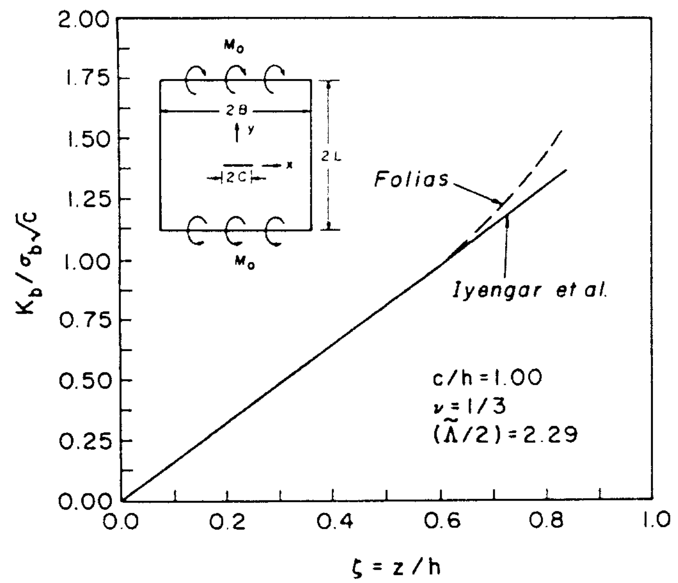

Fig. 4. The stress intensity factor versus z/h for crack to thickness ratio of $c/h = 1.00$.

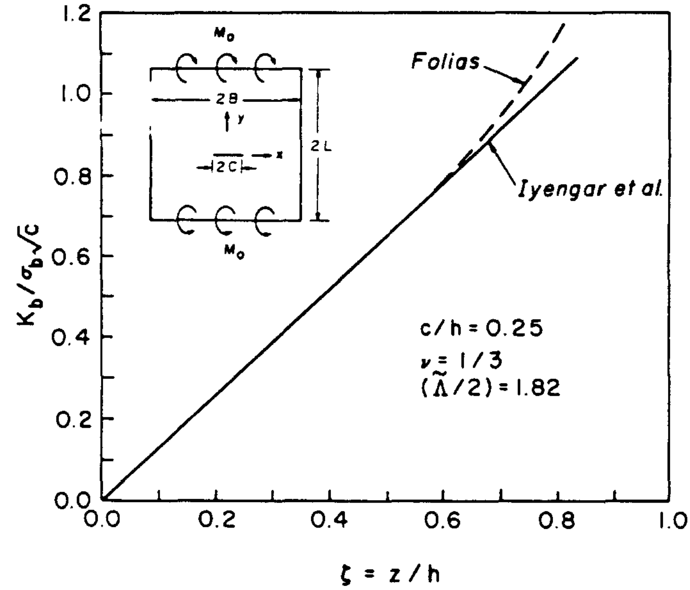

Fig. 5. The stress intensity factor versus z/h for crack to thickness ratio of $c/h = 3.00$

as given by eqns (82)–(87), is plotted in Figs 3, 4 and 5 for a Poisson's ratio of $\nu = 1/3$ and various crack to thickness ratios.†

Recently, the problem of a plate with finite plate dimensions was independently investigated by Iyengar *et al.* (1988). The authors in their analysis use the same method for the construction of the solution with the exception that the details are carried out in cylindrical coordinates. In fact, one may extract the solution in cylindrical coordinates directly from the general solution of this paper (see Appendix B). This establishes, therefore, the parallel character of the two solutions. A comparison of the analytical results [eqns (82)–(87)] with those of Iyengar *et al.* (1988) shows an excellent agreement up to 60–70% of the plate thickness. Beyond this range, the analytical results begin to deviate, for, as was pointed out previously, they represent only the dominant term of the z-dependency. In fact, if one takes into account the remaining terms,‡ the results tend towards those obtained by Iyengar *et al.*, which are compatible with those found by the author in the case of the hole (Folias *et al.*, 1985).

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† In order to compare our results with those of Iyengar *et al.* (1988), which are applicable to a plate with finite dimensions of $L/B = 0.5$, we have multiplied our present results by a factor of 1.19 (see Isida, 1966). Also note that $\tilde{\Lambda} = 1.19\Lambda$.

‡ By solving numerically the system (67) and (70).

APPENDIX A

Fourier series expansions

$$1 = \frac{2}{h}\sum_{n=0}^{\infty}\frac{(-1)^n}{\alpha_n}\cos(\alpha_n z) \qquad -h < z < h.$$

$$z = \frac{2}{h}\sum_{n=0}^{\infty}\frac{(-1)^n}{\alpha_n^2}\sin(\alpha_n z) \qquad -h \leqslant z \leqslant h.$$

$$h^2 - z^2 = \frac{4}{h}\sum_{n=0}^{\infty}\frac{(-1)^n}{\alpha_n^3}\cos(\alpha_n z) \qquad -h \leqslant z \leqslant h.$$

$$\sin(\beta_\nu z) = \frac{2\beta_\nu}{h}\cos(\beta_\nu h)\sum_{n=0}^{\infty}\frac{(-1)^n}{(\alpha_n^2-\beta_\nu^2)}\sin(\alpha_n z) \qquad -h \leqslant z \leqslant h.$$

$$z\cos(\beta_\nu z) = \frac{2}{h}[\cos(\beta_\nu h) - \beta_\nu h\sin(\beta_\nu h)]\sum_{n=0}^{\infty}\frac{(-1)^n}{(\alpha_n^2-\beta_\nu^2)}\sin(\alpha_n z) + \frac{4}{h}\beta_\nu^2\cos(\beta_\nu h)\sum_{n=0}^{\infty}\frac{(-1)^n}{(\alpha_n^2-\beta_\nu^2)^2}\sin(\alpha_n z) \qquad -h \leqslant z \leqslant h.$$

APPENDIX B

Alternate form of the general solution

The displacement and stress fields due to bending may also be written in the following more convenient form.

(i) *The displacement field*:

$$u^{(c)} = -z\left(\frac{\partial I_1}{\partial x} + |y|\frac{\partial I_2}{\partial x}\right) + \frac{z^3}{3}\left(\frac{2m-1}{m-1}\right)\frac{\partial^2 I_2}{\partial x\,\partial y}$$
$$-\sum_{\nu=1}^{\infty}\frac{1}{\beta_\nu}\frac{\partial H_\nu}{\partial x}[(2m-2-m\cos^2(\beta_\nu h))\sin(\beta_\nu z) + m\beta_\nu z\cos(\beta_\nu z)]\cos(\beta_\nu h)$$
$$+\sum_{n=0}^{\infty}\frac{1}{\alpha_n}\frac{\partial H_n}{\partial y}\sin(\alpha_n z)\sin(\alpha_n h)$$

$$v^{(c)} = \pm\left\{z\left(I_2 + \frac{\partial I_1}{\partial y} + |y|\frac{\partial I_2}{\partial y}\right) - \frac{2m-1}{m-1}\frac{z^3}{3}\frac{\partial^2 I_2}{\partial y^2}\right.$$
$$+\sum_{\nu=1}^{\infty}\frac{1}{\beta_\nu}\frac{\partial H_\nu}{\partial y}[(2m-2-m\cos^2(\beta_\nu h))\sin(\beta_\nu z) + m\beta_\nu z\cos(\beta_\nu z)]\cos(\beta_\nu h)$$
$$\left.+\sum_{n=0}^{\infty}\frac{1}{\alpha_n}\frac{\partial H_n}{\partial x}\sin(\alpha_n z)\sin(\alpha_n h)\right\}$$

$$w^{(c)} = I_1 - \frac{2m}{m-1}h^2\frac{\partial I_2}{\partial y} + \frac{1}{m-1}z^2\frac{\partial I_2}{\partial y} + |y|I_2$$
$$+\sum_{\nu=1}^{\infty}H_\nu[(m-2+m\cos^2(\beta_\nu h))\cos(\beta_\nu z) + m\beta_\nu z\sin(\beta_\nu z)]\cos(\beta_\nu h)$$

(ii) *The stress field*:

$$\frac{\sigma_{xx}^{(c)}}{2G} = z\left[\frac{\partial^2 I_1}{\partial y^2} - \frac{2}{m-1}\frac{\partial I_2}{\partial y} + |y|\frac{\partial^2 I_2}{\partial y^2} - \frac{2m-1}{m-1}\frac{z^2}{3}\frac{\partial^3 I_2}{\partial y^3}\right]$$
$$-2\sum_{\nu=1}^{\infty}\beta_\nu H_\nu\cos(\beta_\nu h)\sin(\beta_\nu z)$$
$$-\sum_{\nu=1}^{\infty}\frac{1}{\beta_\nu}\frac{\partial^2 H_\nu}{\partial x^2}[(2m-2-m\cos^2(\beta_\nu h))\sin(\beta_\nu z) + m\beta_\nu z\cos(\beta_\nu z)]\cos(\beta_\nu h)$$
$$+\sum_{n=0}^{\infty}\frac{1}{\alpha_n}\frac{\partial^2 H_n}{\partial x\,\partial y}\sin(\alpha_n z)\sin(\alpha_n h)$$

$$\frac{\sigma_{yy}^{(c)}}{2G} = z\left[\frac{2m}{m-1}\frac{\partial I_2}{\partial y} - \frac{\partial^2 I_1}{\partial y^2} + \frac{2m-1}{m-1}\frac{z^2}{3}\frac{\partial^3 I_2}{\partial y^3} - |y|\frac{\partial^2 I_2}{\partial y^2}\right]$$
$$-2\sum_{\nu=1}^{\infty}\beta_\nu H_\nu\sin(\beta_\nu z)\cos(\beta_\nu h)$$

$$- \sum_{\nu=1}^{\infty} \frac{1}{\beta_\nu} \frac{\partial^2 H_\nu}{\partial y^2} [(2m-2-m\cos^2(\beta_\nu h))\sin(\beta_\nu z) + m\beta_\nu z\cos(\beta_\nu z)]\cos(\beta_\nu h)$$

$$- \sum_{n=0}^{\infty} \frac{1}{\alpha_n} \frac{\partial^2 H_n}{\partial x \partial y} \sin(\alpha_n z)\sin(\alpha_n h)$$

$$\frac{\sigma_{zz}^{(c)}}{2G} = -m \sum_{\nu=1}^{\infty} \beta_\nu H_\nu [\cos^2(\beta_\nu h)\sin(\beta_\nu z) - \beta_\nu z\cos(\beta_\nu z)]\cos(\beta_\nu h)$$

$$\frac{\tau_{xy}^{(c)}}{G} = \pm\left\{2z\left(\frac{\partial^2 I_1}{\partial x\partial y} + \frac{\partial I_2}{\partial x}\right) - \frac{2}{3}\left(\frac{2m-1}{m-1}\right)z^3 \frac{\partial^3 I_2}{\partial x \partial y^2} + 2z|y| \frac{\partial^2 I_2}{\partial x \partial y}\right.$$

$$+2 \sum_{\nu=1}^{\infty} \frac{1}{\beta_\nu} \frac{\partial^2 H_\nu}{\partial x \partial y} [(2m-2-m\cos^2(\beta_\nu h))\sin(\beta_\nu z) + m\beta_\nu z\cos(\beta_\nu z)]\cos(\beta_\nu h)$$

$$\left. - \sum_{n=0}^{\infty} \frac{1}{\alpha_n}\left(\frac{\partial^2 H_n}{\partial y^2} - \frac{\partial^2 H_n}{\partial x^2}\right)\sin(\alpha_n z)\sin(\alpha_n h)\right]$$

$$\frac{\tau_{yz}^{(c)}}{G} = \pm\left\{\frac{2m}{m-1}(h^2 - z^2)\frac{\partial^2 I_2}{\partial y^2}\right.$$

$$+2m \sum_{\nu=1}^{\infty} \frac{\partial H_\nu}{\partial y} [\sin^2(\beta_\nu h)\cos(\beta_\nu z) - \beta_\nu z\sin(\beta_\nu z)]\cos(\beta_\nu h)$$

$$\left. + \sum_{n=0}^{\infty} \frac{\partial H_n}{\partial x}\cos(\alpha_n z)\sin(\alpha_n h)\right\}$$

$$\frac{\tau_{xz}^{(c)}}{G} = -\frac{2m}{m-1}(h^2 - z^2)\frac{\partial^2 I_2}{\partial x \partial y}$$

$$-2m \sum_{\nu=1}^{\infty} \frac{\partial H_\nu}{\partial x} [\sin^2(\beta_\nu h)\cos(\beta_\nu z) - \beta_\nu z\sin(\beta_\nu z)]\cos(\beta_\nu h)$$

$$+ \sum_{n=0}^{\infty} \frac{\partial H_n}{\partial y}\cos(\alpha_n z)\sin(\alpha_n h)$$

where we have assumed that I_1 and I_2 are 2-D harmonic functions and that H_ν and H_n satisfy the 2-D equations

$$\frac{\partial^2 H_\nu}{\partial x^2} + \frac{\partial^2 H_\nu}{\partial y^2} - \beta_\nu^2 H_\nu = 0$$

and

$$\frac{\partial^2 H_n}{\partial x^2} + \frac{\partial^2 H_n}{\partial y^2} - \alpha_n^2 H_n = 0.$$

Notice that in cylindrical coordinates, the equation for H_ν reads

$$\frac{\partial^2 H_\nu}{\partial r^2} + \frac{1}{r}\frac{\partial H_\nu}{\partial r} + \frac{1}{r^2}\frac{\partial^2 H_\nu}{\partial \phi^2} - \beta_\nu^2 H_\nu = 0,$$

the solution of which is

$$H_\nu = \sum_{k=0}^{\infty} \{A_k I_{k+1/2}(\beta_\nu r) + B_k K_{k+1/2}(\beta_\nu r)\}\, e^{i(k+1/2)\phi}$$

which solution is compatible with that reported by Iyengar *et al.* (1988).